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Self-Financed Duration Matching Portfolios: An Exercise in Commercial Bank Asset-Liability Management

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Abstract

The purpose of this paper is to describe a student exercise in asset-liability management. Given the funding opportunities of a commercial bank, the objective of the exercise is to determine the optimal allocation of bank funds so that changes in interest rates do not adversely affect the value of bank assets relative to the value of bank liabilities. The exercise demonstrates that the dynamic nature of asset-liability management in practice is a direct result of the local properties of duration and convexity measures. Techniques of dynamic asset-liability allocation are indicated by the solutions to the asset-liability management problem. The exercise provides an experience in the calculation of optimal funds allocations and the analysis of the performance of the portfolios created using a spreadsheet based simulation.

I. Introduction

Viewing the balance sheet of a commercial bank as a portfolio of claims on future cash flows, the current market value of bank assets minus liabilities or net worth is a function of the rates of interest used to discount future cash flows to the present. In this environment, the net worth of a bank is at risk to adverse changes in market discount rates. Asset-liability management concerns the optimal allocation of funds in a commercial bank asset/liability portfolio so that changes in market rates of interest do not adversely affect the present value of net worth. The purpose of this paper is to describe a student exercise in asset-liability management. The exercise focuses on the characteristics of the optimal allocation of bank funds so that changes in interest rates do not adversely affect the value of bank assets relative to the value of bank liabilities. A review of five popular commercial banking textbooks – Fraser, Gup, and Kolari (2001), Gardner and Mills (2000), Gup and Kolari (2005), Hempel and Simonson (1999), Koch and MacDonald (2015), and Sinkey (2002) - reveals that student exercises requiring explicit calculation of optimal funds allocation in the context of asset-liability management are not offered nor is a solution to the optimal funds allocation problem presented. Several exercises appearing in these textbooks ask students to describe the general kind of transactions that would change the balance sheet in a way that reduces the overall interest rate risk exposure. Exercises requiring explicit calculations of optimal allocations and the analysis of the dynamic nature of asset-liability management are not presently offered in these textbooks. The main objectives of the exercise are discussed in section I. In section II the setup and solution to the asset-liability management problem is described. The student exercises are described and implications of the resulting solutions are discussed in section III. The final section concludes with comments on the implementation of the exercise in the classroom.

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II. Objectives of the Exercise

The main objective of the exercise described in this paper is to analyze the asset-liability management problem in commercial banking that focuses on structuring the balance sheet so that interest rate changes do not erode net worth.[3] To accomplish this objective it is necessary to understand how interest rate changes affect the market value of a bank's assets relative to the market value of that bank's liabilities. The operative concept here is relative values. For example, if interest rates increase, then the present (market) value of both assets and liabilities decrease, however, if the present value of liabilities decrease more than the present value of assets, then the market value of the bank's net worth will clearly increase. Alternatively, if rates rise and asset values fall more than liability values, then the market value of the bank's net worth is eroded. The effect of interest rate changes on the value of assets relative to liabilities is the central concern of asset-liability management.

The form of interest rate innovation encountered in practice determines the corresponding change in a bank's assets and liabilities. Though interest rate changes along the spot rate term structure may assume an infinite array of contortions in practice, an explicit exercise in asset-liability management must begin with a manageable form of interest rate shift. The exercise described here allows for any initial shape of the term structure and focuses on the effects of an instantaneous additive rate shock to spot rates of interest. The magnitude of the additive rate shock is assumed to be random and applied uniformly across the term structure of spot rates. The additive rate shock thus induces a uniform and parallel shift to the term structure. Though the level of the rate shock is assumed to be random, the general displacement of the term structure in terms of a parallel shift is deterministic. Ingersoll, Skelton, and Weill (1978), among others, show that deterministic rate shocks admit arbitrage opportunities. This discovery has led to models of term structure innovations that are arbitrage free and the development of strategies to hedge the risk of holding fixed income securities in such arbitrage free environments.[4] This is certainly an important line of research on fixed income securities, however, the study of arbitrage free term structure models is generally beyond the scope of most commercial banking courses particularly at the undergraduate level. Nevertheless, the study of asset-liability management in the context of deterministic rate shocks can provide an introduction to the problem that is readily accessible to students possessing only an elementary understanding of calculus.[5]

The mere existence of arbitrage opportunities does not eliminate the benefits to be gained by studying the consequences of deterministic rate shocks. Indeed, the ability to construct portfolios that can profit from deterministic rate shocks implies that the form of interest rate risk most relevant in practice arises from changes in default and maturity risk premiums across the term structure.[6] Furthermore, the importance of changes in premiums as a source of risk in fixed income security markets provides a rationale for the study of arbitrage free term structure innovations. Similarly, the study of the asset-liability management problem in the context of deterministic rate shocks applies fundamental results from fixed income research leading to greater appreciation of the practical importance of advanced term structure theory. Understanding the asset-liability management strategy in a deterministic rate shock environment provides a foundation from which more complex models can be approached. Truly any deterministic shock to the term structure will provide a similar learning experience. The additive rate shock is the most elementary and, thus, is a natural point of departure. The asset-liability management exercise discussed below employs mostly standard results from the duration and immunization literature.[7] Duration is a function of the first derivative of the present value of a cash flow stream with respect to an interest rate shock. In general, the first derivative of a function expresses the slope of a line tangent to the function. A line tangent to a nonlinear function is, of course, only an approximation to the local shape of the function.

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[3] Many factors may cause the net worth of a commercial bank to change including a change in the risk characteristics of a bank's business activities or changes in general market conditions. The traditional asset-liability management problem considered here focuses on interest rate effects in a partial equilibrium environment where the affect of other factors is held constant.

[4] The literature is extensive. Some examples are Cox, Ingersoll and Ross (1992), Heath, Jarrow and Morton (1992), and Ho and Lee (1986).

[5] The basic result from calculus used in the exercise is the derivative of a polynomial, that is, for $u = c(1 + x)^n$ find $du/dx = nc(1 + x)^{n-1}$ where c is a constant.

[6] Bond market analysis is replete with discussion of spreads between the bond yields of different maturities. For example, the bond market analysis by Breifing.com available at the yahoo.com website discusses innovations in yield spreads nearly every day.

[7] See, for example, Bierwag (1987).

Because the present value of a cash flow stream is a nonlinear function of an interest rate shock, duration is thus a local measure of the change in the present value of the cash flow stream with respect to an infinitesimal shock to the rate of interest.[8] For finite rate shocks, duration is an approximate measure of present value changes implying that in practice asset-liability management must be a dynamic strategy because following a rate shock it is necessary to reallocate funds to optimally hedge against subsequent rate shocks. Through the use of explicit examples, the exercise discussed below illustrates the dynamic nature of asset-liability management in practice by showing the degree of approximation that arises in application of duration measures.The second derivative of a cash flow stream with respect to an interest rate shock yields a measure generally known as convexity.[9] It follows from the above discussion that convexity is also an approximate local measure for finite rate shocks. The exercise demonstrates that the local convexity of net worth with respect to interest rate shocks depends critically on the allocation of funds into different cash flow maturities. In particular, a fixed allocation of funds cannot produce a global convexity in net worth because the present value of both assets and liabilities approach zero in the limit as the discount rate increases and, hence, net worth approaches zero in the limit. Thus, net worth cannot be a convex function of an interest rate shock in the large or globally. Explicit examples used in the exercise demonstrate the local nature of convexity and indicate the general characteristics of a dynamic strategy of funds allocation to hedge against a sequence of interest rate shocks.

III. Setup of Problem and Solution

A. Preliminaries

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A list of variables and notation used in the remainder of the paper is displayed in Table I. The future (asset) cash inflow, A_i, is received with certainty at time T_i (i = 1, 2, 3, 4). The total present value of cash inflows is $A = \sum_i A_i(1+r_{Ai}+\lambda)^{-T_i}$ where r_{Ai} is the spot discount rate for A_i, and λ is the instantaneous additive rate shock. Let $W_{Ai} = A_i(R_{Ai})^{T_i}(A)^{-1}$ denote the present value weight of (asset) cash inflow A_i where $R_{Ai} = (1+r_{Ai})^{-1}$ represents the discount factor given $\lambda = 0$. The modified duration of future (asset) cash inflows is

$$D_A = (-A)^{-1}\,\partial A/\partial\lambda = \sum_i T_i\, W_{Ai}\, R_{Ai} \qquad (1)$$

The same notation is used for (liability) cash outflow variables except that L replaces the A's. The modified duration of future (liability) cash outflows is

$$D_L = (-L)^{-1}\,\partial L/\partial\lambda = \sum_i T_i\, W_{Li}\, R_{Li} \qquad (2)$$

B. Discussion

The functional form of duration is a natural result of historical development. Macaulay (1938) originally expressed duration as a "weighted average maturity" in a seminal paper examining the effect of yield changes on bond prices. If we consider A as representing the price of a bond and let $r_{Ai} = y\ \forall\ i$ represent the bond yield then $(-A)^{-1}\partial A/\partial = (1+y)^{-1}\sum_i T_i W_{Ai}$ and the measure of duration originally proposed by Macaulay, $\sum_i T_i W_{Ai}$, is clearly a "weighted average maturity". Macaulay duration is implied by an additive rate shock but is limited in application to bond yields and/or flat term structures wherein it is possible to factor out the term $(1+y)^{-1}$ from the summation appearing in the derivative of the bond price. Nevertheless, Macaulay's "weighted average maturity" concept is so compelling in its simplicity and convenient in application that it has become a standard measure in fixed income analysis. Building on Macaulay's original work, researchers have developed different formulations of duration, each of which are consistent with a specific form of interest rate shock.[10] Note that in the formulation of section A above, the spot discount rates r_{Ai} and r_{Li} are not necessarily equal for a given i. Furthermore, the spot discount rates are not necessarily equal for different maturities T_i. The spot rate curves for borrowing and lending can therefore assume any initial shape and the traditional Macaulay duration must be modified by including the discount factors R_{Ai} and R_{Li}, respectively, in equations (1) and (2).[11] The modification of Macaulay duration used in this paper allows for any initial shape of the term structure of interest rates.

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[8] More precisely, duration approximates the absolute value of the percentage change in the present value of a cash flow stream (Crack and Nawalkha (2001)). The issue is discussed further below.

[9] See, for example, Bierwag (1987).

[10] ibid.

[11] The modification to the traditional Macaulay duration can be avoided by expressing the present value of cash flows using continuous compound rates of interest. See, for example, Crack and Nawalkha (2001).

Three time points provide a simple solution to the asset-liability problem. Four time points provide sufficient detail to illustrate how the asset portfolio should be reallocated following a shift in the term structure. Additional time points can certainly be included but will only complicate the ensuing calculations without any discernable benefit to instruction in the basic asset-liability management problem.[12]

C. Definition of the General Problem and Solution

The asset-liability management problem can be stated as follows. Given the liability present value weights $W_{Li}>0$ (i = 1, 2, 3, 4) and asset present value weights $W_{Ai}>0$ (i = 3, 4), choose asset allocations (weights) $W_{Ai}>0$ (i = 1, 2) so that net worth, $N = A–L$, is a local minimum with respect to the additive rate shock λ. Obviously, a solution to the problem requires two independent expressions that determine the asset allocation weights W_{Ai} (i = 1, 2). First, since the asset weights must sum to one, we can immediately express W_{A1} as

$$W_{A1} = 1 - W_{A2} - W_{A3} - W_{A4} \qquad (3)$$

In order to obtain W_{A2}, we use the first order condition for a local maximum or minimum value of net worth with respect to the rate shock λ. The first order condition is

$$\partial N/\partial\lambda \,\Big|_{\lambda = 0} = 0 \qquad (4)$$

In order to obtain a solution to (4) above, the relative values of A and L must be specified. For simplicity we adopt the self-financing condition where $A=L$ which allows for the most straightforward interpretation of the results unencumbered algebraically by an arbitrarily chosen position of initial positive net worth, $A–L>0$. In particular, the self-financing condition, $A=L$, implies that the denominators of the asset and liability weights, W_{Ai} and W_{Li}, are equal $\forall\ i$ which in turn simplifies the expression of $\partial N/\partial\lambda$. Given the self-financing condition, a solution to the first order condition (4) implies the following duration matching condition.

$$D_A = D_L \Rightarrow W_{A2} = \{D_L - T_1(1 - W_{A3} - W_{A4})R_{A1} - T_3W_{A3}R_{A3} - T_4W_{A4}R_{A4}\}[T_{A2}R_{A2} - T_1R_{A1}]^{-1} \qquad (5)$$

The asset weight condition (3) and the self-financing duration matching condition (5) determine the asset weights for i = 1, 2 and given the remaining asset and liability allocations the initial balance sheet is completely specified. The problem solution specifying the asset allocations is summarized in Table II.

Table II. Asset Allocation Weights
Asset Weight Condition: $W_{A1} = 1 - W_{A2} - W_{A3} - W_{A4}$
Self-Financing Duration Matching Condition: $\partial N/\partial\lambda \,\big|_{\lambda = 0} = 0$ and $A = L \Rightarrow D_A = D_L$ $\Rightarrow W_{A2} = \{D_L - T_1(1 - W_{A3} - W_{A4})R_{A1} - T_3W_{A3}R_{A3} - T_4W_{A4}R_{A4}\}[T_{A2}R_{A2} - T_1R_{A1}]^{-1}$

D. Discussion of the Solution to the General Funds Allocation Problem

Crack and Nawalkha (2001) discuss the confusion that can arise due to the factors $(-A)^{-1}$ and $(-L)^{-1}$ appearing in equations (1) and (2) above and show that duration is appropriate in the analysis of the relationship between the rate shock λ and changes in A, L, and $N = A–L$ measured as a percent of $A=L$ prior to the rate shock or $\lambda=0$. The percentage changes or instantaneous returns in A, L, and N resulting from finite λ are expressed, respectively, as $\Delta A/A$, $\Delta L/L$, and $\Delta N/A$. A main purpose of the exercise is to illustrate the characteristics of the asset-liability allocations that result in the local convexity of $\Delta N/A$ with respect to changes in λ.[13] The second order condition for $\Delta N/A$ a minimum is expressed in equation (6).[14]

$$(A)^{-1}\partial^2 N/\partial\lambda^2 \,\Big|_{\lambda = 0} = \textstyle\sum_i T_i(T_i-1)[W_{Ai}(R_{Ai})^2 - W_{Li}(R_{Li})^2] > 0 \qquad (6)$$

Examination of the second order condition (6) shows that, due to the $T_i(T_i–1)$ term, convex $\Delta N/A$ results from an allocation of assets across a wide dispersion of maturities relative to the dispersion of liability allocations. In general, if R_{Ai} and R_{Li} do not differ significantly and the asset allocations are "spread out" over a wider range of maturities than the liability allocations, then $\Delta N/A$ is a minimum at $\lambda=0$. An allocation of funds resulting in local convexity of $\Delta N/$

[12] Additional time points could be useful, for example, in the study of bond portfolios or swap agreements, however, these topics are beyond the scope of this paper where the focus is on the basic problem of asset-liability management in commercial banking.

[13] Note that local convexity in ΔN/A is necessary and sufficient for local convexity in N.

[14] As shown below, ΔN/A can be a global minimum at λ = 0 but can only be locally convex at λ = 0.

A with respect to the rate shock λ is the property that produces a "free lunch" because an increase in N following any finite λ represents a risk free profit. Since the self-financing condition implies that the asset/liability portfolio is costless, any risk free profit is clearly a "free lunch". Whether or not a self-financed duration matching portfolio can generate a riskless profit in practice is, of course, an empirical question. Bierwag (1987) and Ho (1990) review empirical tests of alternative duration measures and find that the "modified" Macaulay duration used in this paper performs well against alternative measures of duration. The exercise described in the next section illustrates the general characteristics of the asset allocations that satisfy the second order condition for $\Delta N/A$ a minimum.

IV. The Student Exercise

A. The General Organization of the Exercise

The student exercise can be separated into three parts. First, the introduction of the notation and problem setup (section II above) must be presented before assigning specific problems that require student calculations. The two remaining parts of the exercise involve student problem solving by calculation and here a pedagogical approach is suggested in order to first focus on the major implications of the solution before proceeding to more complex forms of the problem. The second part of the exercise involves solutions to a simplified form of the general problem discussed in section II above that readily yields to straightforward analysis of the characteristics and implications of optimal asset allocations. The suggested simplification is to limit the number of cash flow allocations. The third part of the exercise introduces the complexities of additional maturities. The general performance of optimal fund allocations employing additional maturities is not materially different from the solution to the simplified problem.

B. Getting Started With a Simplified Form of the Asset-Liability Management Problem

The simplified form of the problem developed in this section allows for non-flat term structure of spot rates. The ultra-simple case of two asset cash flows, a single liability cash flow, and flat term structure is solved in Appendix A. Let $W_{A3} = W_{A4} = 0$ and let the liability allocation be concentrated at a single maturity, say $W_{L4} = 1$. A solution is guaranteed provided that the liability maturity is between the two asset maturities ($T_1 < T_4 < T_2$).[15] If the liability maturity is chosen as the midpoint of the asset maturities, $T_4 = (T_1+T_2)$ /2, then the asset weights equal approximately one half.[16] If r_{A1}, r_{A2}, and r_{L4} do not differ significantly, then the second order condition expressed in equation (6) is clearly satisfied due to the $T_i(T_i-1)$ term and since the liability allocation is concentrated at a maturity between the two asset maturities. These simplifications allow relative ease in the calculation of $\Delta A/A$, $\Delta L/L$, and $\Delta N/A$ for different values of the rate shock 8 thereby allowing us to focus on the implications of results before proceeding to more complex forms of the problem with allocations distributed across all four maturities. Given the cash flow maturities ($T_1 < T_4 < T_2$), the students are instructed to calculate W_{A1}, W_{A2}, and the values of the instantaneous return variables $\Delta A/A$, $\Delta L/L$, and $\Delta N/A$ for a domain of the rate shock 8 and then to graph $\Delta A/A$, $\Delta L/L$, and $\Delta N/A$ as functions of 8. Displayed in Figures 1 is a graph of ΔA and $\Delta L/L$ showing $\Delta A > \Delta L/L$ for the chosen parameters. Figure 2 displays the local convex property of $\Delta N/A$ given the parameter values chosen.

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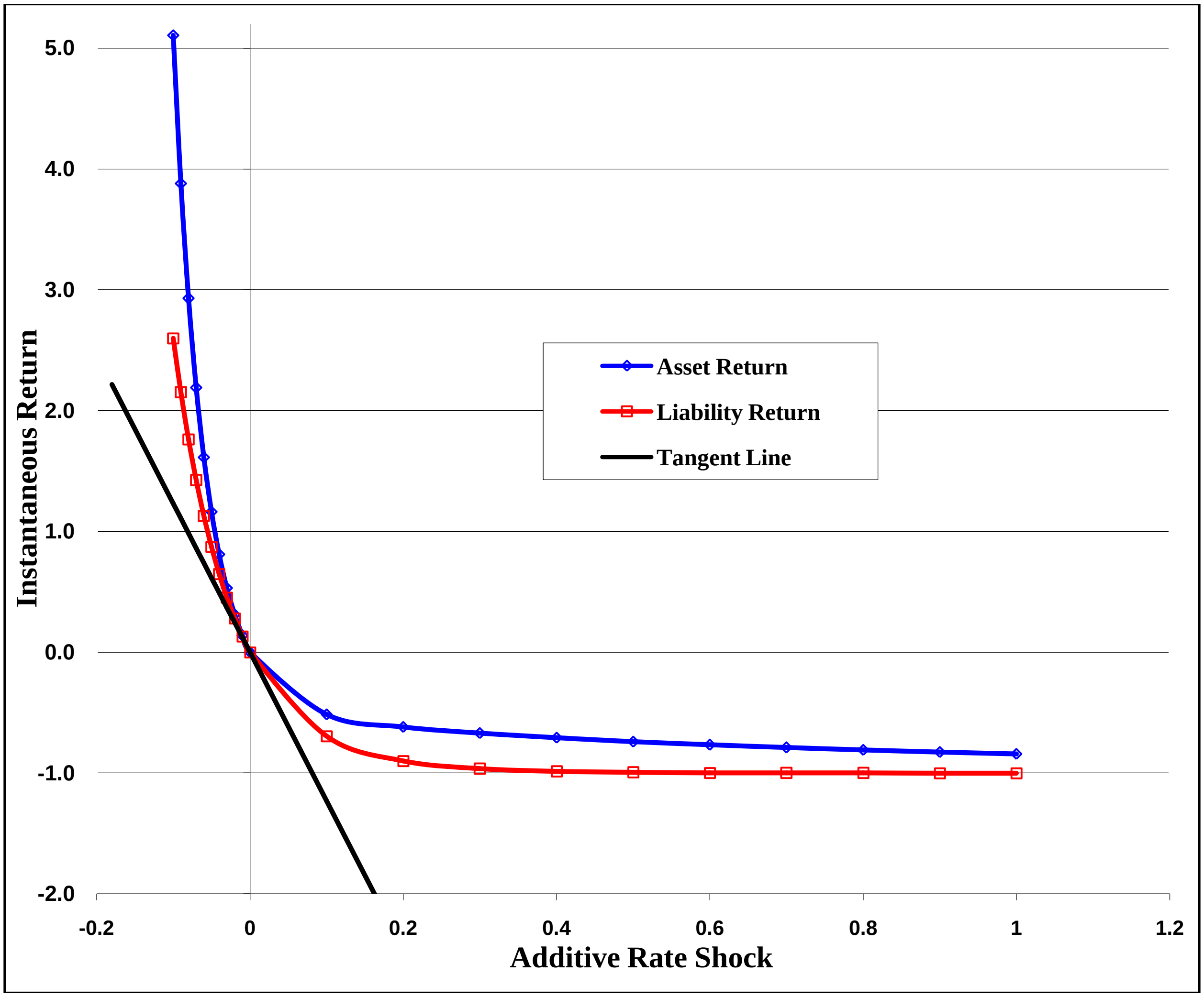
[15] The index i on the maturity variable is used in the notation instead of subscripting the present values of the cash flows directly by T. In this way T_i is not necessarily increasing in i thereby providing the flexibility to employ a range of initial cash flow allocations across different maturities. The use of different initial cash flow allocations and different cash flow maturities to provide examples of the possible different shapes of the $\Delta N/A$ curve is central to the purpose of the exercise.

[16] The weights equal one-half exactly if $r_{Ai} = r_{Li}\ \forall$ i.

Figure 1. Instantaneous return on *L* and *A* due to the additive rate shock *8*. The figure corresponds to asset allocations $W_{A1} = W_{A2} = 0.5$ occurring at maturities $T_1 = 2$ and $T_2 = 30$, the single liability allocation $W_{L4} = 1$ with maturity $T_4 = 16$, and $r_{Ai} = r_{Li} = 0.3\ \forall i$.

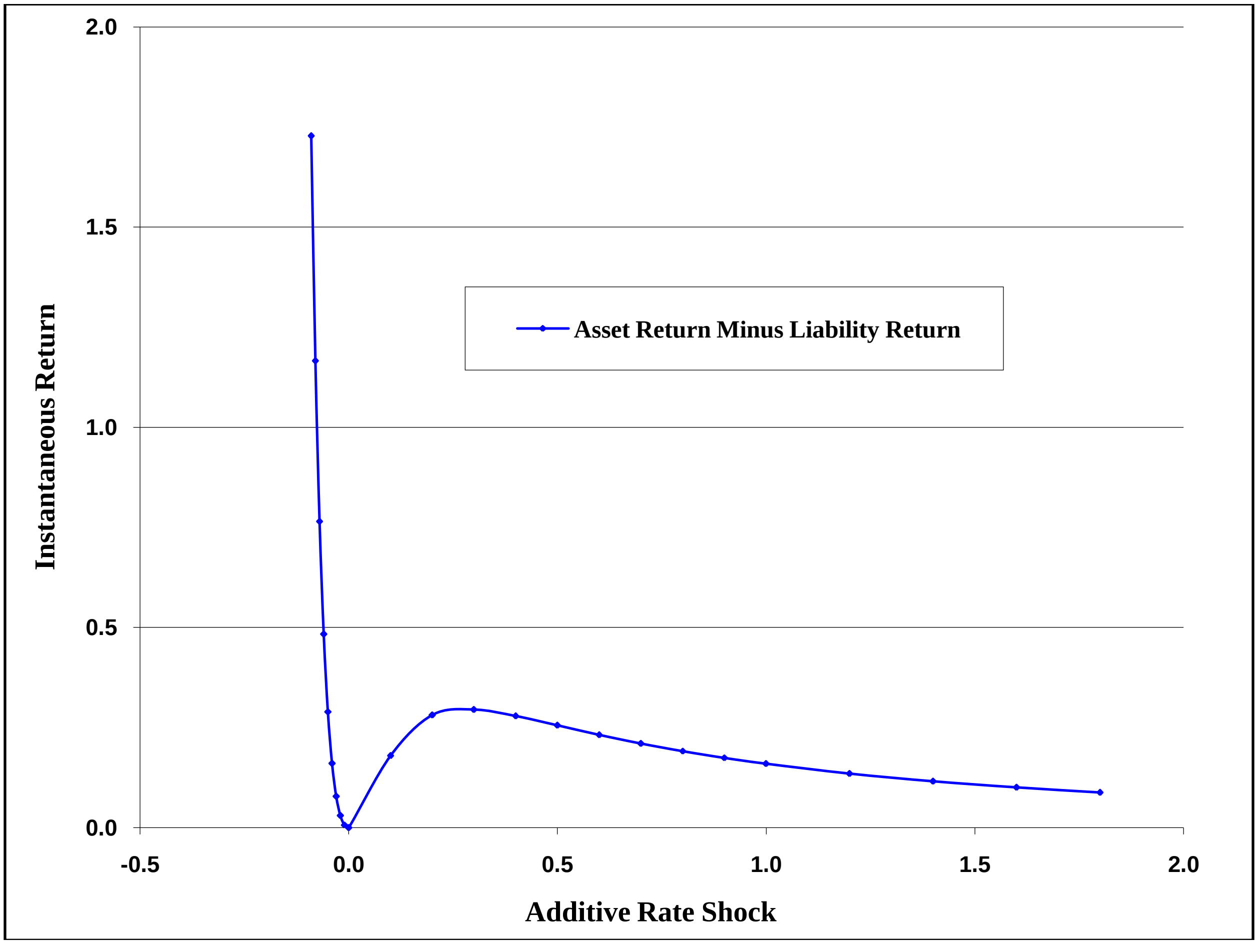


C. Discussion of the Solution to the Simplified Form of the Asset-Liability Management Problem

The absolute value of the slope of the tangent line in Figure 1 is equal to the duration of *A* and *L* evaluated at λ=0 and clearly illustrates the approximation to the non-linear curves of $\Delta A/A$ and $\Delta L/L$. Figure 1 also shows the relationship between the instantaneous returns in *A* and *L* and the net contributions each make to the value of $\Delta N/A$ displayed in Figure 2. Note that $\Delta N/A = (\Delta A - \Delta L)/A = \Delta A/A - \Delta L/L$ due to the self-financing condition $A = L$. The net contribution of $\Delta A/A$ and $\Delta L/L$ to $\Delta N/A$ is non-symmetric around λ=0 where $\Delta N/A$ is strictly increasing (convex) for λ<0 but for λ>0 the $\Delta N/A$ curve is initially convex and then becomes concave as λ increases. A point of inflection thus occurs on the $\Delta N/A$ curve in the positive domain of λ indicating that the self-financing and duration-matching conditions are not sufficient to achieve global convexity of $\Delta N/A$. The limiting value of $\Delta N/A$ for λ>>0 occurs because, as shown in Figure 1, $\Delta A/A$, $\Delta L/L \rightarrow -100$ percent as $\lambda \rightarrow +\infty$ which, of course, occurs because *A*, $L \rightarrow 0$ as $\lambda \rightarrow +\infty$. Note that for "small" finite λ,[17] $\Delta A/A > \Delta L/L$ does not correspond to convex $\Delta N/A$.

[17] The instantaneous rate shock 8 is "small" in the sense that ΔN/A remains in the local convex or concave region of its range near 8=0.

Figure 2. Instantaneous return on assets *(A)* minus the instantaneous return on liabilities *(L)* due to the additive rate shock λ. The figure corresponds to asset allocations $W_{A1} = W_{A2} = 0.5$ occurring at maturities $T_1 = 2$ and $T_2 = 30$, the single liability allocation $W_{L4} = 1$ with maturity $T_4 = 16$, and $r_{Ai} = r_{Li} = 0.3\ \forall i$.



Instead the net difference $\Delta A/A – \Delta L/L$ must be strictly increasing for convex $\Delta N/A$. As shown in Figure 1, the net difference $\Delta A/A – \Delta L/L$ is not strictly increasing in the positive domain of λ. It follows that $\Delta N/A$ can be a global minimum at λ=0 but can only be locally convex near λ=0. A dynamic strategy for funds allocation is indicated by the results of the initial exercise. In general, $\partial A/\partial\lambda < 0$ and $\partial L/\partial\lambda < 0$ since a present value will decrease as the rate of interest increases. Note, however, that D_A and D_L are positive due the factors $(–A)^{-1}$ and $(–L)^{-1}$, respectively, in expressions (1) and (2). Due to the self-financing condition, N=0 when λ=0 and since $\Delta N/A$ is locally convex near λ=0, a "small" positive rate shock (λ >0) results in $\Delta N/A$>0 and the slope of the $\Delta N/A$ curve in Figure 2 is positive. Note that $\Delta N/A$>0 implies $\Delta A – \Delta L$>0. But since both ΔA, ΔL < 0 when λ>0, it follows that the present value of liabilities (L) must decline more than the present value of assets (A) in order for net worth to increase following a "small " positive λ. Therefore, if the present values of both assets and liabilities decline given λ>0 but simultaneously $\Delta N = \Delta A – \Delta L > 0$, then $\partial L/\partial\lambda < \partial A/\partial\lambda < 0$ implying that $D_L > D_A$ and the duration matching condition is no longer satisfied since, otherwise, the $\Delta N/A$ curve is not convex in the (local) positive domain of λ. The dynamic strategy following a small positive rate shock thus requires an increase in D_A in order to reestablish the duration matching condition and is accomplished by reallocating funds from the present value of A_1 into the present value of A_2.[18] Intuitively, a "small" positive rate shock produces a large decline in the present value of A_2 relative to the decline in the present value of A_1 that must be replenished in order to reestablish the duration matching condition.

[18] Recall that in the initial simplified problem $T_1 < T_2$.

A similar argument indicates that a negative rate shock results in $D_A > D_L$ and the duration matching condition is reestablished by reallocating funds from the present value of A_2 into the present value of A_1. In general, following a positive rate shock ($\lambda > 0$) the duration matching condition is reestablished by increasing Macaulay's "average maturity" of asset cash flows and, conversely, following a negative rate shock ($\lambda < 0$) the duration matching condition is reestablished by decreasing Macaulay's "average maturity" of asset cash flows.

D. Completing the Exercise Using Allocations at All Four Maturities

In the third part of the exercise, students calculate asset weights for the complete problem with asset and liability allocations spread across all four maturities. At this point a spreadsheet was introduced that calculates and graphs $\Delta N/A$ for a domain of finite shocks *8* and the asset weights determined by the student. Using several initial allocations the students are introduced to examples of $\Delta N/A$ as local convex and local concave functions of *8*. The second order condition for a minimum in N becomes obvious with the different initial asset and liability allocations. As shown in equation (5) above, satisfying the second order condition for N a minimum depends on the sum over i of $T_i(T_i-1)$ weighted by the factor $[W_{Ai}(R_{Li})^2 - W_{Li}(R_{Li})^2]$. In particular, given relatively uniform liability weights, $W_{Li} \approx 1/4\ \forall i$, and letting $T_1 < T_2 < T_3 < T_4$, then the relative value of the asset weights will determine the corresponding convexity or concavity of N if the interest rate spreads, $r_{Ai} - r_{Li}$, are not unreasonably large. For example, if $W_{A1} + W_{A4} >> W_{A2} + W_{A3}$ then the asset allocations are "spread out" across the maturities more than the uniformly distributed liability allocations and N will be locally convex with respect to changes in *8*. Alternatively, if $W_{A1} + W_{A4} << W_{A2} + W_{A3}$ then the liability allocations are "spread out" more than the asset allocations and N will be locally concave.

V. Conclusions

The three main stages of the exercise can be completed in less than two hours spread over several class meetings. The problem setup and introduction to the initial simplified problem can be accomplished in less than one hour. Discussion of the students' results from working the initial problem and introduction to the full problem with allocations across all four maturities usually takes half an hour. Finally, 30 minutes spread over several class meetings is sufficient to review the students' results from calculating the allocations in several problems exhibiting convex and concave $\Delta N/A$.

Overall the exercise provides some relief from the many descriptive and non-analytic topics covered in a typical commercial banking course. The students benefit from hands on experience in calculating optimal asset-liability allocations that hedge against interest rate shock and from the analysis of the portfolios they create. Many students have shown a sense of accomplishment after calculating the correct allocation weights and seeing first hand the shape of the resulting ΔN□ $/A$ curve just as theory has predicted. The exercise also illustrates the importance of modeling innovations in the term structure. In particular, changes in risk and term premiums as a source of risk for fixed income portfolios becomes evident after examining the performance of optimal funds allocations in the presence of additive rate shocks. Unexpected twists in the term structure caused by changes in risk and term premiums is an important source of risk that can not be hedged effectively by using traditional duration matching strategies. Obviously more complex models of the term structure must be employed to successfully hedge against such non-deterministic rate shocks. If the exercise is presented carefully, students can become intrigued with the analysis and develop a greater appreciation of modeling term structure innovation and the application to asset-liability management.

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Appendix A. Flat spot rate term structure

Two-Asset Self-Financed Duration-Matching Portfolio

There exist three pure discount bonds that pay X_i at time t_i (i = 1, 2, 3) with present values: X_1 $(1+R)^{-t1}$,X_2 $(1+R)^{-t2}$, and X_3 $(1+R)^{-t3}$. The initial rate is assumed to be R compounded each period. Without loss of generality, let $t_1 < t_2 < t_3$.

Self-Financing Condition

First, let $L = X_2\,(1+R)^{-t2}$ and choose X_2 such that $X_2\,(1+R)^{-t2} = 1$; i.e. let $X_2 = (1+R)^{+t2}$ so that $L = 1$. We create a liability by selling L and promising to repay the amount $X_2 = (1+R)^{+t2}$ at time t_2 and we use the borrowed funds ($L = 1$) to purchase the portfolio $A = X_1\,(1+R)^{-t1} + X_3\,(1+R)^{-t3}$. We are investing the same amount that we borrow, so the "out of pocket" expense is zero, i.e. self-financing. In general, the self-financing condition requires that $L = A$ where $L = 1$ is the amount we borrow (our liability) and A is our asset investment which is equal to the present value of the investment portfolio. Note that since L =1, we also have A = 1 by the self financing condition $L = A$. Furthermore, after setting $L = 1 = A$, the self financing condition implies that:

$$X_1 = (1+R)^{+t1}\,[1 - X_3\,(1+R)^{-t3}] \quad (1)$$

Duration-Matching Condition

The duration of our liability is $D_L = \{-(1+R)/L\}(\mathrm{d}L/\mathrm{d}R) = t_2$.
The duration of our asset portfolio is $D_A = \{-(1+R)/A\}(\mathrm{d}A/\mathrm{d}R) = t_1\,X_1\,(1+R)^{-t1} + t_3\,X_3\,(1+R)^{-t3}$.
The duration-matching condition requires that $D_A = D_L$ which implies that:

$$t_3\,X_3\,(1+R)^{-t3} = t_2 - t_1\,X_1\,(1+R)^{-t1} \quad (2)$$

Substituting (1) into (2) we obtain:

$$X_3 = (1+R)^{+t3}(t_2 - t_1)/(t_3 - t_1) \quad (3)$$

Equations (3) and (1) define the self-financing duration-matching portfolio weights X_1 and X_3 completely.

Example

Suppose that R = 6%, t_1 = 4, t_2 = 6, and t_3 = 10. Using (3) we calculate:
$X_3 = (1.06)^{10}(6-4)/(10-4) = 0.596949$.
Then using (1) we calculate:
$X_1 = (1.06)^4[1-0.596949(1.06)^{-10}] = 0.841651$
Using the spreadsheet posted on Blackboard, $A-L$ is a minimum at R = 6%.