

Downside Loss Aversion and Portfolio Growth

Jivendra K. Kale¹ & Arnav Sheth²

Abstract

Optimizing over power-log utility functions allow for the inclusion of downside loss aversion, a broader range of investor preferences, and account for higher-order moments like skewness and kurtosis in the optimization process. We implement multi-period power-log optimization (PLO) with annual rebalancing on a portfolio consisting of a treasury security, the S&P500 index and a call option on the index. PLO results in higher geometric average realized returns with lower tail risk, and lower standard deviation than mean-variance efficient portfolios with the same ex-ante expected returns. It also provides better downside protection against large, negative return surprises, such as the down markets in 2002 and 2008.

1. Introduction

Ever since Markowitz (1953) introduced the idea of a mean-variance efficient portfolio, his technique has been the dominant method for portfolio selection. Its failings are well known. It works well when asset returns are approximately normal, but when asset returns are skewed and have fat tails, mean and variance are insufficient for specifying investor preferences³. These two moments alone do not allow for the contribution of positive skewness in securities like call options or bonds, which many investors find desirable. In fact, mean-variance optimization results in treating the upper tail returns as suboptimal since they produce a higher variance of return (see Pedersen 2001). Additionally, mean-variance analysis can lead to hidden risk in a portfolio when asset returns have fat tails because of its assumption of normality (see Leland 1999).

Several methods have been developed for portfolio construction with assets that have skewed and fat-tailed returns, but they are focused on controlling lower tail risk. They trade off mean return for a portfolio against some measure of downside risk, such as semi-variance (Markowitz 1959), or lower partial moments (Bawa 1978, Fishburn 1977, Jarrow and Zhao 2006). Semi-variance fails to account for higher moments such as kurtosis and skewness, while lower partial moments assume a risk-neutral investor on the upside. Other methods that account for downside risk like value at risk, (RiskMetrics 1996), or conditional value at risk (Basak and Shapiro 2001, Rockafellar and Uryasev 2000) are not generated from investor preferences, but enter as constraints in the utility maximization problem. Also, while these methods are successful at constraining for downside risk, none take advantage of upper tail gains. As stated in Leland (1999), on semi-variance and value at risk "These approaches are not grounded in capital market equilibrium theory and may themselves spuriously identify superior or inferior managerial performance." The power-log utility function (Kale, 2006) combines behavioral finance with multi-period portfolio theory to resolve all of the above issues.

¹ Ph.D., CFA, St. Mary's College of California

² Ph.D., St. Mary's College of California

³ Patton (2004) showed that knowledge of both skewness and asymmetric dependence leads to economically significant gains, in particular, with no shorting constraints. Harvey, Liechty, Liechty, and Müller (2010) proposed a method for optimal portfolio selection involving a Bayesian decision theoretical framework that addresses both higher moments and estimation error. They suggested that incorporating higher-order return distribution moments in portfolio selection is important.

It is a two-segment function, where the utility of gains is modeled with a log utility function and the utility of losses is modeled with a power utility function with power less than or equal to zero⁴. It combines the maximum growth characteristics of the log utility function on the upside, with the scalable downside protection characteristics of the power function on the downside.

It is defined as,

$$U = \ln(1+r) \quad \text{for } r \geq 0 \quad (5)$$

$$= \frac{1}{\gamma}(1+r)^\gamma \quad \text{for } r < 0$$

where,

r portfolio return

γ power, is less than or equal to 0

The following three characteristics make power-log utility functions more representative of investor preferences than say, the quadratic utility functions of Markowitz (1952) or the other downside loss aversion techniques described above. First, optimizing over these utility functions requires the specification and use of the entire joint distribution of asset returns. All the moments and cross-moments of the distribution including mean, variance, skewness, kurtosis of asset returns, and correlation, coskewness and cokurtosis between asset returns are taken into account. If some of the assets have return distributions that are skewed, the optimization process produces portfolios with the positively skewed return distributions that reflect the growth maximization and loss aversion properties. Other studies, such as Harvey, Liechty, Liechty and Muller (2010), and Hitaj, Martellini and Zumbrano (2010), account for a subset of the moments and cross-moments of the asset return distribution, but do not take into account *all* the moments and cross-moments. Second, power-log utility functions are continuously differentiable across the entire range of returns. The slope of the log function that is used for gains is 1 when return is zero, and the slope of the power function that is used for losses is also 1 when the return is zero, for all powers less than or equal to zero. Thus, even though power-log utility functions are steeper for negative returns than they are for positive returns, they do not have a kink at a return of zero, and that allows the development of fast optimization algorithms for portfolio selection. The basis of the algorithm used for this study is a nonlinear mathematical programming algorithm based on an accelerated conjugate direction method developed by Best and Ritter (1976), and has a superlinear rate of convergence⁵.

Third, across the entire domain of returns, the power-log utility functions are characterized by increasing utility, and diminishing marginal utility as returns increase. Thus, they conform partially to the Tversky and Kahneman (1991, p. 1039) postulates of reference dependence and loss aversion, and the S-shaped utility function from prospect theory. Power-log utility functions exhibit Tversky and Kahneman's postulate of diminishing sensitivity for gains, but they do not exhibit diminishing sensitivity for losses, which models risk seeking behavior that is represented by a convex function for losses. Power-log utility functions are characterized by an *increasing* sensitivity to losses as the size of losses increases, which represents risk averse behavior that is modeled with a concave function. As a result, power-log utility functions are concave for the entire domain and model risk averse investor behavior across the board. While prospect theory is appropriate as a descriptive model for explaining speculative behavior, it is not appropriate as a normative model of investor preferences for constructing portfolios. The experiments in support of prospect theory, such as those described in Kahneman and Tversky (1979), are designed as gambles where only a small fraction of an individual's wealth is at stake and do not apply to investors who are investing significant portions of their wealth, for example for their retirement funds. Cremers, Kritzman and Page (2005) find that investors with S-shaped preferences are attracted to kurtosis as well as negative skewness, which is contrary to the wellknown investor preference for positive skewness in returns. Thus, when compared to Kahneman and Tversky's S-shaped utility function, the power-log utility functions' increasing sensitivity to losses is a better representation of investor preferences for constructing portfolios. In fact, for power-log utility functions the utility associated with a 100% loss is negative infinity.

⁴ The log utility function is a special case of the power utility function. As the power gamma goes to zero in the limit, the utility function reverts to log utility (Grauer and Hakansson, 1982).

⁵ The authors thank Financimetrics Inc. for access to its optimization algorithms.

The powerlog utility functions will not allow for the selection of a portfolio where a 100% loss has a positive probability, and this is not true of prospect theory's S-shaped utility function. Thus, power-log utility functions work very well as a normative model for representing investor preferences. Figure 1 shows an example of a power-log utility function with downside powers of -1 and -15. The utility function for gains is the log utility function, and utility functions for losses are the power function with powers of -1 and -15. Thus, investors can vary the level of downside protection they build into their portfolios by changing the downside power and lower values of the downside power represent greater aversion to losses since the penalty for losses increases.

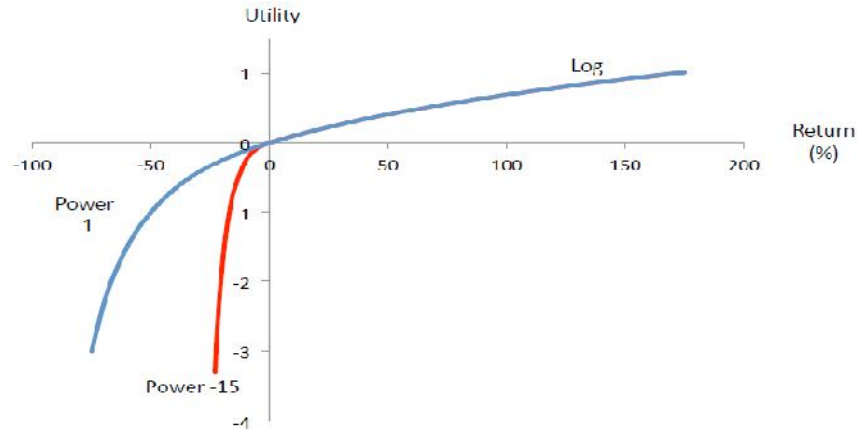


Figure 1: Power-Log Utility Functions with Downside Power -1 and -15

In this paper, we use empirical data from 1996 through 2009 for a treasury security, the S&P500 index and a call option on the index, to test power-log optimization with annual rebalancing over multiple periods. We rebalance the portfolio using a distribution of non-clairvoyant, ex-ante returns and have the PLO maximize expected utility to match the expected returns from a meanvariance optimization on the same distribution. For virtually the entire range of investor preferences from high risk to low risk, optimal power-log portfolios deliver higher geometric average realized returns with lower tail risk and lower standard deviation than mean-variance efficient portfolios that have the same ex-ante expected returns. We also find that optimal powerlog portfolios provide much better downside protection against large, negative return surprises than the matched mean-variance efficient portfolios, for example, in the years 2002 and 2008 when the market was down substantially. In down markets, it is common for portfolio managers to write call options to profit from the premiums when the calls expire worthless. We also find, surprisingly, that it is *always* suboptimal for portfolio managers to write call options when they believe the market will be down. Our results show that they are better off going long on the call option and short on the underlying index itself.

2. Methodology

The expected utility criterion developed by Von Neumann and Morgenstern (1944), and Savage (1964) gives us the following one-period optimization problem for selecting assets weights:

Maximize

$$E(U) = \sum_s p_s U_s \tag{2}$$

where,

s scenario s, and the summation is over all scenarios

ps probability of scenario s

Us utility in scenario s, based on the portfolio return in the scenario, rs,

where the utility function is the log function in Equation 1

The portfolio return in scenario s is calculated as a weighted average of the returns to the assets in the portfolio,

$$r_s = \sum_i w_i r_{is} \quad (3)$$

where,

r_i asset i , and the summation is over all assets in the portfolio
 w_i investment weight of asset i in the portfolio
 r_{is} return to asset i in scenario s

For our empirical test over multiple periods, we construct optimal portfolios using power-log utility functions and rebalance them once a year. The assets in the portfolio are a one-year treasury security, the S&P500 index, and the closest to the money call option on the S&P500 index with approximately one year to expiration. The data is for the years 1996-2009, which includes periods with wide swings in the stock market⁶. For each year in the sample period, the portfolio is purchased on the December expiration date of the call option and sold on the December expiration date of the following year, which is the rebalancing date. In other words, portfolio rebalancing is synchronized with the December expiration dates of the call options. For example, for the first holding period an optimal portfolio is constructed and purchased on December 20, 1996, the expiration date in December 1996; it is held for about a year and sold on December 19, 1997, the expiration date in December 1997, when a new optimal portfolio is constructed and purchased for the following year. To construct the optimal portfolio at the beginning of this one-year holding period, we select the closest to the money call option that expires on December 19, 1997, for inclusion in the portfolio. The one-year constant maturity yield on a treasury as of December 20, 1996 is used to calculate the riskless return for the holding period. To calculate portfolio values we use market prices for the S&P500 index at the beginning and end of the holding period. For the call option price, we use the average of the closing bid and ask on the first day of the holding period, and the expiration value on the last day of the holding period, because the expiration value gives us a more reliable valuation of the call than the market price on the expiration date.

To construct an optimal portfolio with a power-log utility function, we need the joint distribution of returns for all assets that can be included in the portfolio. The key distribution for our empirical test is the distribution of returns for the S&P500 index. For this, we use a deterministic, simulated standard normal distribution to represent the S&P500 index. We transform this distribution and assign it a mean of 10%, and a standard deviation equal to the Black-Scholes implied volatility of the selected calls, which is also the reason for selecting calls that are closest to the money at our rebalancing date. The calculated Black-Scholes implied volatility gives us a market forecast for the standard deviation of return for the holding period. Since there is no universally accepted forecast for the mean of the distribution, we use 10% as the mean log return for the index, which is its approximate geometric mean for the postwar period. We examined the shape of the distribution of annual S&P500 returns and performed three separate EDF tests on annual S&P 500 returns from 1950 through 2011: the Lilliefors test, the Jarque-Bera test and the χ^2 test, with the null hypothesis that the return distribution is lognormal, and got p-values of 0.7107, 0.3371 and 0.6797 respectively. Thus, we cannot reject the hypothesis of lognormality at the 5% level of significance. Assuming that the distribution of S&P500 returns is approximately lognormal, also allows us to use the Black-Scholes option pricing model to calculate the implied volatility for the index from the price of the closest to the money call option at the beginning of each holding period⁷.

To avoid the problem of isolated randomly generated points in the tails of the simulated S&P500 return distribution for a given holding period, we use deterministic simulation instead of Monte Carlo simulation. We start by generating one million equal probability points by dividing the domain of the lognormal return distribution into one million intervals of equal probability, and then calculate the median for each interval. Next we use a clustering algorithm, Kale (2011), to reduce the one million points to 10,000 clusters, where each cluster's probability corresponds to the number of points in the cluster. Using one million equal probability intervals to start with, gives us sufficient and consistent representation in the tails of the return distribution, and reducing them to 10,000 points makes simulation more efficient.

⁶ The S&P500 index data is from yahoo.com, and the options data is from CSIData.

⁷ Note that while we find that the annual S&P500 returns are approximately lognormal, this is not true for weekly, or daily S&P500 returns. As the frequency of returns increases, the return distribution becomes markedly leptokurtic, and then neither the Black-Scholes implied volatility nor a lognormal distribution assumption would be appropriate for simulating future returns.

To generate the call option returns that correspond to the simulated S&P500 index returns for a given holding period, we use the market value for the index at the beginning of the holding period, and the 10,000 simulated index returns to generate 10,000 index values for the end of the holding period. Next we calculate the expiration values for the closest to the money call option based on these index values and the strike price, and then use these expiration values and the call price at the beginning of the holding period to calculate the 10,000 call option returns that correspond to the 10,000 S&P500 index returns. We combine the return distributions for the treasury, the S&P500 index and the call option on the index, to create a joint distribution for the three assets with 10,000 observations for each holding period, and use it for optimization with power-log utility functions. Although the power-log optimization algorithm we have used for this study can create portfolios with long or short positions in any of the assets, we put in a “no short sales” constraint on the S&P500 index and the call option in order to make it easier to interpret the results. To provide a context for evaluating the performance of the optimal power-log portfolios we construct mean-variance efficient portfolios with matched ex-ante returns. While it can be argued that mean-variance analysis is inappropriate for constructing portfolios containing options since their return distributions have significant higher moments, it is the most widely used and familiar methodology for portfolio construction and does provide an interesting comparison. In the sections that follow we compare the portfolio compositions and risk and return characteristics of portfolios constructed with the two techniques.

3. Results

We start by constructing optimal portfolios with the log utility function, by setting the downside power to zero in the power-log utility function. The resulting portfolios are the riskiest in our sets of portfolios. Next we construct the matched mean-variance efficient portfolio for each period, such that its ex-ante expected return equals the ex-ante expected return for the optimal power-log portfolio. As seen in Table 1, the mean-variance efficient portfolios are very different from the optimal powerlog portfolios in all but one of the periods⁸, and most of them have zero exposure to the call option. The positive skewness of the call option returns and its higher upper tail returns are treated as higher risk in the mean-variance optimization (MVO). To evaluate the performance of the optimal portfolios, we use realized returns for the three assets. Table 1 shows realized returns for the optimal Power-Log portfolios for downside power zero, and matched mean-variance efficient portfolios. The best performance for both sets of portfolios was in period 1, where coincidentally the matched portfolios had the same composition, and the realized return for them was 68.88%, which is far above the ex-ante expected return of 27.97% for that period. This was a result of the stellar market performance in 1997, which produced an S&P500 index return of 28.43% in period 1, the best return for any period. In theory, the best return for the optimal Power-Log portfolios should be higher than that for the mean-variance efficient portfolios because Power-Log optimization preserves the positive skewness in asset returns, but for this 1996-2009 data set they turned out to be the same.

⁸ Coincidentally, the asset weights for the mean variance efficient portfolios are identical to those for the optimal power-log portfolios in December 1996 and 2006.

Table 1: Realized Returns for Optimal Power-Log and Matched Mean-Variance Efficient Portfolios (Downside Power 0)

Holding Period	Start Date	End Date	Ex-Ante Expected Return (%)	Realized Return (%)	
				Optimal Power-Log	Mean-Variance Efficient
1	12/20/1996	12/19/1997	27.97	68.88	68.88
2	12/19/1997	12/18/1998	20.94	38.80	43.61
3	12/18/1998	12/17/1999	20.82	23.43	32.09
4	12/17/1999	12/15/2000	17.81	-17.06	-13.08
5	12/15/2000	12/21/2001	18.95	-20.16	-21.32
6	12/21/2001	12/20/2002	32.86	-39.08	-61.24
7	12/20/2002	12/19/2003	32.48	42.80	54.75
8	12/19/2003	12/17/2004	68.40	32.79	65.19
9	12/17/2004	12/16/2005	58.57	1.25	34.19
10	12/16/2005	12/15/2006	43.38	37.17	38.17
11	12/15/2006	12/21/2007	41.27	-15.25	-15.25
12	12/21/2007	12/19/2008	29.46	31.03	96.21
13	12/19/2008	12/18/2009	29.32	24.22	41.21
Geometric Average			33.25	6.16	-12.26
Ending Value of \$1				2.18	0.18
Standard Deviation				33.53	50.71
Skewness				0.01	-1.01

The two worst returns for both sets of portfolios were in periods 6 and 12, reflecting the big unanticipated market losses of 2002 and 2008 respectively. In period 6 the S&P500 index had a return of -20.39%, and in period 12 it had a return of -38.32%. In both periods the optimal Power-Log portfolios suffered a significantly smaller loss than the matched mean-variance efficient portfolios. The difference was particularly dramatic in period 12, where the optimal Power-Log portfolio had a return of -31.03%, while the mean-variance efficient portfolio had a return of -96.21% and came close to bankruptcy. Portfolios selected by using Power-Log optimization should never go bankrupt (Rubinstein (1991)), and this is supported by our results, although Rubinstein's analysis does not account for discrete rebalancing and differences between ex-ante and ex-post return distributions. This particular period was also interesting in showing us how much more sensitive mean-variance optimization is to small changes in inputs, than Power-Log optimization. If we replace the one-year constant maturity treasury yield as our riskless rate with a three-month constant maturity treasury yield, the optimal Power-Log portfolio's realized return for this period changes from -31.03% to -33.87%, while the matched mean-variance efficient portfolio's return changes from -96.21% to -100.00%, i.e., bankruptcy!

Table 1 also shows the geometric average returns over the thirteen periods. They are 6.16% for optimal Power-Log portfolios, versus -12.26% for mean-variance efficient portfolios. While the difference in geometric average returns is large, the difference in the ending value of a Dollar is dramatic, \$2.18 for optimal Power-Log portfolios versus \$0.18 for mean-variance efficient portfolios. The standard deviation of return is not a particularly good measure for risk for skewed return distributions, but what makes the Table 1 numbers interesting is that the sample standard deviation of return for optimal Power-Log portfolios, 33.53%, is substantially smaller than the sample standard deviation of return for mean-variance efficient portfolios, 50.71%, even though the mean-variance efficient portfolios were constructed specifically to minimize variance in each period, given the expected return for that period. This is a result of smaller losses suffered by optimal Power-Log portfolios during unanticipated market declines. The substantial negative skewness of -1.01 for mean-variance efficient portfolios versus 0.01 for optimal Power-Log portfolios also reflects that observation. Figure 3 shows histograms of returns for the two sets of portfolios, and highlights the smaller lower-tail risk of optimal Power-Log portfolios. Figure 2 shows histograms for the optimal power-log portfolio returns for downside power zero, and matched mean-variance efficient portfolio returns. Optimal power-log portfolios have far smaller lower tail risk than the matched mean-variance efficient portfolios, and thus provide far better downside protection against large unanticipated market declines.

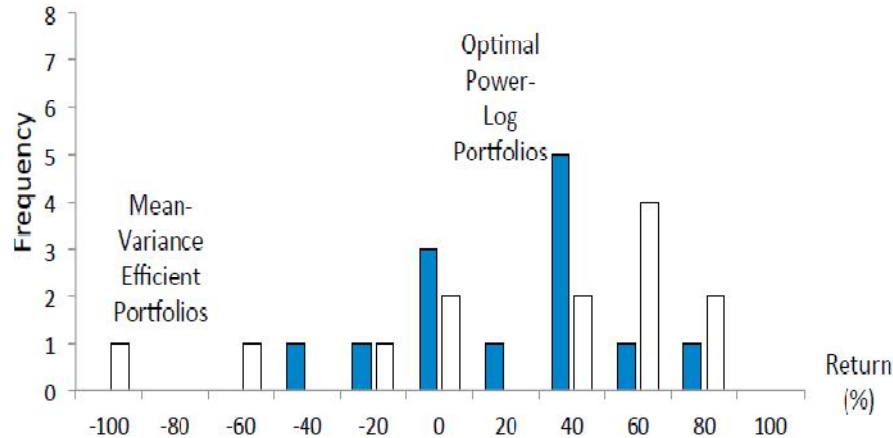


Figure 2: Realized Returns (1996-2009)

It is tempting to compare the arithmetic average returns of the time series of realized returns for the two portfolio construction methods, since there are standard tests for comparing means. For example, a paired differences test could be a good test, since it is not affected by the strong correlation between the two series. However, any test using arithmetic averages is inappropriate in a multi-period context where we compound returns to judge performance over multiple periods, and the distribution of returns is changing from one period to the next. For example, the average value of the realized return for optimal power-log portfolios minus the realized return for mean-variance efficient portfolios is -2.06%, which might suggest that the performance for optimal power-log portfolios is worse than for mean-variance efficient portfolios, but the geometric average return for optimal power-log portfolios is 6.16% versus -12.26% for the meanvariance efficient portfolios.

While some investors (Samuelson, 1971) may accept the risk associated with portfolios constructed with the log utility function, which is a special case of the power-log utility function, the risk is unacceptable for the vast majority of investors. The smallest investment in the call option was 17.85% for the optimal power-log portfolio in holding period 4, and the largest investment was 46.72% in holding period 9. These large investments in derivatives carry a lot of risk, which can be reduced by reducing the downside power (making it more negative), thus increasing the penalty for losses. We redid the optimizations for several different downside powers and Table 2 summarizes the realized returns for optimal power-log portfolios for five downside powers, 0 through -50, and the matching mean-variance efficient portfolios. As the downside power decreases, the penalty for losses increases and the optimal power-log portfolios become more conservative. The minimum return increases from -39.08% for the downside power of zero, to -2.08% for a downside power of -50, and tail risk drops dramatically.

Table 2: Range, Standard Deviation and Skewness of Realized Returns (1996-2009)

Optimal Power-Log Realized Returns (%)					
Downside Power	Min.	Geo. Avg.	Max.	Standard Deviation	Skewness
0.00	-39.08	6.16	68.88	33.53	0.01
-0.60	-30.20	6.38	58.53	27.49	0.09
-3.00	-16.73	5.90	40.88	17.77	0.26
-9.00	-8.32	5.19	28.68	11.30	0.52
-50.00	-2.08	4.35	18.90	6.45	1.09

M-V Efficient Realized Returns (%)					
Downside Power	Min.	Geo. Avg.	Max.	Standard Deviation	Skewness
0.00	-96.21	-12.26	68.88	50.71	-1.01
-0.60	-79.37	-0.37	58.53	42.09	-1.01
-3.00	-50.04	4.64	40.88	27.13	-1.00
-9.00	-29.32	5.12	28.68	16.64	-0.93
-50.00	-12.26	4.37	18.90	8.40	-0.33

Comparing the optimal power-log and mean-variance efficient portfolios in Table 2, we see that the minimum return for the optimal power-log portfolios is better than that for the matching mean-variance efficient portfolios for every downside power. The optimal power-log portfolios have lower tail risk across the board. On the upside, both sets of portfolios had their best return in holding period 1 and the compositions were identical for each matched pair of optimal portfolios for all downside powers. Table 2 also shows that the sample standard deviation of realized returns is much lower for optimal power-log portfolios than for mean-variance efficient portfolios across the board, for both risky and conservative portfolios. The consistency of this result is surprising, since in theory, mean-variance optimization is supposed to produce portfolios with lower standard deviation for a given level of expected return. As mentioned before, this is a result of large unanticipated market declines in some periods, which implies that ex-ante and ex-post return distributions are significantly different in those periods. Power-log optimization performs better in this type of environment and produces portfolios that have less lower tail risk, and a smaller standard deviation of return than matched mean-variance efficient portfolios. Looking at the geometric average realized return in Table 2 we see that it is positive for optimal power-log portfolios for each downside power, rising initially from 6.16% for a downside power of zero, to 6.38% for a downside power of -0.6, and then declining steadily to 4.35% for a downside power of -50. According to theory, the downside power of zero, which corresponds to the log utility function, should produce the maximum growth portfolio, i.e., the portfolio with the highest geometric average return. That would have been true if the ex-ante and ex-post distributions were the same, but that is not the case here since there are always unanticipated changes in the economic environment. Large unanticipated negative returns are likely to have lowered the geometric average return for downside power zero to less than that for downside power -0.6, which provides better protection against large losses. Of course, we might find that for longer sample periods than the one used here, portfolios constructed with a downside power of zero outperform all other portfolios, as theory predicts.

Comparing the geometric average return for optimal power-log and mean-variance efficient portfolios in Table 2, we see that for medium and high risk portfolios the return is far better for optimal power-log portfolios, but about the same for the most conservative portfolios. Figure 3 extends these results to 34 downside powers from zero to -50, showing the corresponding ending values of a dollar. For the optimal power-log portfolios it rises from \$1.74 for the most conservative portfolios to \$2.24 as the ex-ante expected return increases and then drops a little to \$2.18 for the riskiest set of portfolios. For the mean-variance efficient portfolios it also starts at \$1.74 for the most conservative portfolios, rises to a maximum of \$1.92 for the medium risk portfolios and then appears to fall off a cliff as the portfolios get riskier and ends up at \$0.18 for the riskiest portfolios.

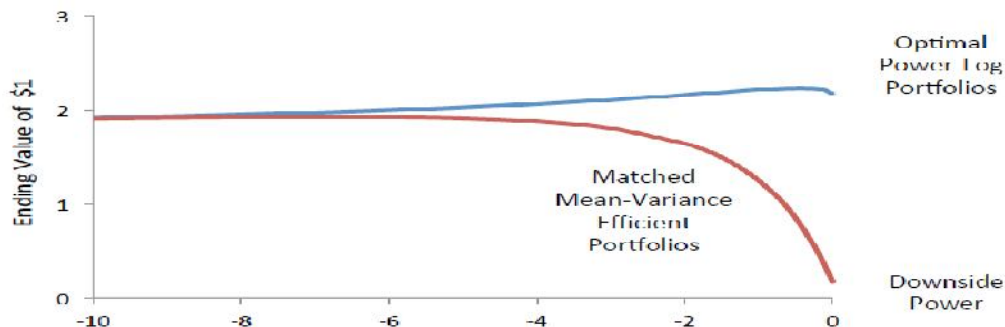


Figure 3: Ending Value of \$1 (1996-2009)

3.1 Call Writing

When expecting down markets, portfolio managers often write calls against the assets they hold in their portfolios. If they are correct, the calls expire out of the money and they reap the call premiums. As power-log optimization picks up return characteristics of options we hypothesize that it should also produce a good call writing strategy for managers in down markets. To test this, we change the assumption for S&P500's expected log return from 10% to -10% to simulate a consistent down market for the joint return distributions for 1996 through 2008. We set the lower bound on the weight of the riskless asset, the S&P500 index and the close to the money call option on the index to -900%, and the upper bound to 900% to permit significant leverage. The other inputs to the optimization remain unchanged.

Table IX shows the weights and returns for the optimal power-log portfolios for downside power -50, and for the matched mean-variance efficient portfolios. None of the optimal power-log portfolios have a short position in the call option in any of the years. Instead, they all have a large short position in the S&P500 index, which is counterbalanced to a large extent by a long position in the call option. The short position in the S&P500 index takes advantage of the negative expected return, while the long position in the call option hedges a big portion of the S&P500 exposure and adds positive skewness to the portfolio's return distribution. In contrast, the meanvariance efficient portfolios have a significant short position in the call option in about half of the years, in addition to a large short position in the S&P500 index in every year; it does not take advantage of the skewness in option returns. We tested this on other more aggressive downside powers as well. This indicates that all investors, aggressive and conservative, are better off taking a short position in the underlying asset along with a long position in the call option, instead of writing calls.

**Table 3: Optimal Portfolios and Realized Returns
(S&P500 Expected Log Return -10%, Downside Power -50)**

Holding Period	Ex-Ante Expected Return (%)	Optimal Power-Log Portfolios			Realized Return (%)
		Riskless	S&P500	Call Option	
1	23.44	307.07	-226.67	19.65	-7.78
2	14.28	213.34	-126.85	13.51	-4.33
3	9.09	168.46	-77.02	8.56	-2.84
4	13.95	206.69	-120.50	13.81	6.26
5	13.54	206.69	-121.56	14.87	11.00
6	5.62	154.91	-60.21	5.30	10.42
7	3.05	131.27	-31.71	3.41	-2.19
8	6.60	178.23	-82.79	4.56	-4.41
9	14.05	247.98	-157.10	9.43	-6.12
10	26.48	362.89	-282.80	19.91	-8.88
11	36.78	452.56	-379.92	27.37	-10.86
12	8.52	177.91	-86.59	8.68	30.37
13	0.71	107.66	-9.17	1.51	-1.02
<i>Geometric Average</i>	13.15				0.21
<i>Ending Value of \$1</i>					1.03
<i>Standard Deviation</i>					11.29
<i>Skewness</i>					1.67
Holding Period	Ex-Ante Expected Return (%)	Mean-Variance Efficient			Realized Return (%)
		Riskless	S&P500	Call Option	
1	23.44	221.34	-118.12	-3.22	-27.99
2	14.28	173.02	-73.93	0.91	-9.30
3	9.09	148.31	-50.22	1.91	-2.61
4	13.95	160.90	-60.26	-0.64	14.08
5	13.54	163.33	-63.81	0.48	16.24
6	5.62	139.71	-40.85	1.14	10.30
7	3.05	127.60	-29.78	2.19	-2.58
8	6.60	153.44	-53.04	-0.40	-4.32
9	14.05	185.10	-81.71	-3.39	-1.70
10	26.48	230.44	-121.08	-9.36	-15.04
11	36.78	254.15	-137.18	-16.97	12.07
12	8.52	160.35	63.51	3.16	26.47
13	0.71	107.34	-9.00	1.67	-0.87
<i>Geometric Average</i>	13.15				0.17
<i>Ending Value of \$1</i>					1.02
<i>Standard Deviation</i>					14.51
<i>Skewness</i>					-0.22

4. Conclusion

Mean-variance analysis has been the standard method for portfolio selection for over half a century, but it penalizes upper tail gains and lower tail losses equally when reducing risk by reducing variance of portfolio return. It works well for assets whose returns have approximately normal distributions, since investor preferences can be completely specified with mean and variance. For portfolios containing assets such as options and bonds, whose return distributions are skewed and have fat tails, mean and variance are insufficient for specifying investor preferences. Methods of portfolio construction that have been developed for these types of portfolios typically focus on controlling downside risk, and are ad-hoc methods that are not based on utility theory. Power-log utility functions combine behavioral finance with multi-period portfolio theory to represent investor preferences realistically, and model the entire range of investor preferences from high-risk maximum growth portfolios to conservative portfolios with a lot of downside protection. Using power-log utility functions to optimize portfolios containing a treasury security, the S&P500 index and a call option on the index, we show that for virtually the entire range of investor preferences, optimal power-log portfolios deliver higher geometric average realized returns with lower tail risk and surprisingly lower standard deviation, than mean-variance efficient portfolios that have the same ex-ante expected returns. We also find that the optimal power-log portfolios provided much better downside protection against the large unanticipated market downturns in 2002 and 2008 than the corresponding mean-variance efficient portfolios. Since it works for portfolios containing assets with normal and non-normal returns, power-log portfolio optimization should work well in all portfolio optimization applications.

References

- Bawa, V. S. (1978). Safety-first, stochastic dominance, and optimal portfolio choice. *Journal of Financial and Quantitative Analysis*, 255-271.
- Best, Michael J., and Klaus Ritter. (1976). A Class of Accelerated Conjugate Direction Methods for Linearly Constrained Minimization Problems. *Mathematics of Computation*, 30(135), 478-504.
- Cremers, Jan-Hein, Mark Kritzman and Sebastien Page. (2005). Optimal hedge Fund Allocations: Do Higher Moments Matter? *The Journal of Portfolio Management*, 31, 3, 70-81.
- Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *The American Economic Review*, 67(2), 116-126.
- Friedman, Milton., and L. J. Savage (1948). The Utility Analysis of Choices Involving Risk. *The Journal of Political Economy*, 56, 4, 279-304.
- Grauer, Robert R., and Nils H. Hakansson. (1982). Higher Return, Lower Risk: Historical Returns on Long-Run, Actively Managed Portfolios of Stocks, Bonds and Bills, 1936-1978. *Financial Analysts Journal*, 38, 39-53.
- Harvey, Campbell R., John C. Liechty, Merrill W. Liechty, and Peter Muller (2010). Portfolio Selection with Higher Moments. *Quantitative Finance*, 10, 5, 469-485.
- Hitaj, Asmerilda, Lionel Martellini, and Giovanni Zambruno. (2012). Optimal Hedge Fund Allocations with Improved Estimates for Coskewness and Cokurtosis Parameters. *The Journal of Alternative Investments*, 14, 3, 6-16.
- Jarrow, R., & Zhao, F. (2006). Downside loss aversion and portfolio management. *Management Science*, 52(4), 558-566.
- Kahneman, Daniel, and Amos Tversky. (1979). Prospect Theory: An Analysis of Decision Under Risk. *Econometrica*, 47, 263-291.
- Kale, Jivendra K. (2006). Growth Optimization with Downside Protection: A New Paradigm for Portfolio Selection. *The Journal of Behavioral Finance*, 7(1), 146-150.
- Kale, Jivendra K. (2011). A Deterministic Simulation Methodology For Evaluating Optimal power-log Utility And Other Portfolios When Asset Returns Are Skewed, Presented at the Fortieth Annual Meeting of the Western Decision Sciences Institute.
- Kelly, J. L. (1956). A New Interpretation of Information Rate. *Bell Systems Technical Journal*, 35, 917-926.
- Leland, Hayne. (1999). Beyond Mean-Variance: Performance Measurement in a Nonsymmetrical World. *Financial Analysts Journal*, January/February 1999, 27-36.
- Markowitz, Harry M. (1952). The Utility of Wealth. *Journal of Political Economy*, 60, 151-158.
- Markowitz, Harry M. (1952). Portfolio Selection. *Journal of Finance*, 8, 77-91.
- Markowitz, Harry M. (1959, 1991). *Portfolio Selection*. Blackwell.
- Markowitz, Harry M. (1976). Investment for the Long Run: New Evidence for an Old Rule. *Journal of Finance*, 31, 1273-1286.
- Patton, A. J. (2004). On the out-of-sample importance of skewness and asymmetric dependence for asset allocation. *Journal of Financial Econometrics*, 2(1), 130-168.
- Pedersen, C. (2001). Derivatives and downside risk. *Derivatives Use, Trading and Regulation*, 7(3), 251-268.
- RiskMetrics – Technical Document, J. P. Morgan, December 17, 1996.
- Rockafellar, R. T., and S. Uryasev. (2000). Optimization of Conditional Value-at-Risk. *Journal of Risk* 2, 21-41.
- Samuelson, Paul A. (1971). The "Fallacy" Of Maximizing The Geometric Mean in Long Sequences of Investing or Gambling. *Proceeding of the National Academy of Science USA*, 68, 2493-2496
- Savage, L. (1964). *The Foundations of Statistics*. Wiley, New York.
- Tversky, Amos, and Daniel Kahneman. (1991). Loss Aversion in Riskless Choice: A Reference-Dependent Model. *Quarterly Journal of Economics*, 106, 1039-1061.
- Von Neumann, John, and Oskar Morgenstern. (1944). *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, New Jersey.